1 Quadratic Equations and Inequalities

1.1 Completing the Square

Lecture 2

1.1.1 Part I

A perfect square is any number or polynomial that can be written as a number squared. The number 25 is a perfect square because $25 = 5^2$, $x^2$ is a perfect square, and $(x - 3)^2$ is a perfect square. The polynomial $x^2 - 6x + 9$ is also a perfect square in that $x^2 - 6x + 9 = (x - 3)^2$. In the last section, a quadratic equation could be solved by the square root method if one side of the equation was a perfect square. In this section, the goal is to find a way to make any quadratic equation solvable using this technique, which is called **completing the square**.

Further, the motivation for solving by completing the square is that it provides a means to solve quadratics that could not be solved. Recall that our current ability required that the quadratic equation be set to zero and factored; however, not all quadratics can be factored. For example, $x^2 - 4x + 1 = 0$ cannot be solved by factoring; however, if the problem is written as $(x - 2)^2 = 3$ it can be found to have solutions of $x = 2 + \sqrt{3}$ and $x = 2 - \sqrt{3}$. The goal of this section is to learn how to solve a quadratic equation by completing the square.

The challenge lies in solving quadratics where the perfect square is not obvious. As a warm-up for the concept, multiply the following perfect squares. The first one is done, the reader should complete the rest.

$$
(x - 3)^2 = (x - 3)(x - 3) \\
= x^2 - 3x - 3x + 9 \\
= x^2 - 6x + 9
$$

1. $(x - 2)^2$  
2. $(x + 2)^2$  
3. $(x - 3)^2$  
4. $(x + 3)^2$  
5. $(x - 4)^2$  
6. $(x + 4)^2$  
7. $(x - 5)^2$  
8. $(x + 5)^2$  
9. $(x - 6)^2$  
10. $(x + 6)^2$  
11. $(x - t)^2$  
12. $(x + t)^2$
The purpose of these problems is to see the pattern. Recall that a quadratic has the form \( ax^2 + bx + c \). Examine numbers 11. and 12. above.

11. \((x-t)^2 = x^2 - 2tx + t^2\) 

and 

12. \((x+t)^2 = x^2 + 2tx + t^2\)

This is the pattern. Notice that the \(b\) term of the difference squared is \(-2t\) and the \(b\) term of the the sum squared is \(2t\). In both perfect squares, \(c = t^2\). These are aligned visually below,

\[
\begin{align*}
ax^2 &+ bx + c \\
x^2 &+ 2tx + t^2
\end{align*}
\]

Once the pattern is understood, instead of expanding \((x-7)^2\) by going through the distribution process, the result is obtained by \(x^2 + 2(-7)x + (-7)^2 = x^2 - 14x + 49\).

This knowledge is also used in the other direction as well, as illustrated in the next example.

**Example.** \(x^2 + 8x + \underline{\phantom{0}} = (x\underline{\phantom{0}})^2\).

The question is to figure what numbers should occupy the two blank lines. Compare the general form of a quadratic, with the current problem.

\[
\begin{align*}
ax^2 &+ bx + c \\
x^2 &+ 2tx + t^2
\end{align*}
\]

Then, \(b = 8\) from the term of \(8x\), which is the result of two times a number: \(8 = 2t\). Finding half of \(8\), \(t = \frac{8}{2} = 4\) is obtained. Use the value of \(t\) to find \(c\). Then, \(c = t^2 = 4^2 = 16\). Thus, the problem can be completed as,

\[
x^2 + 8x + 16 = (x+4)^2
\]

**Workout**. Complete the following perfect squares.

\[
\begin{align*}
&x^2 + 16x + \underline{\phantom{0}} = (x\underline{\phantom{0}})^2 \\
&x^2 - 14x + \underline{\phantom{0}} = (x\underline{\phantom{0}})^2 \\
&x^2 - 10x + \underline{\phantom{0}} = (x\underline{\phantom{0}})^2 \\
&x^2 + 18x + \underline{\phantom{0}} = (x\underline{\phantom{0}})^2 \\
&x^2 - 7x + \underline{\phantom{0}} = (x\underline{\phantom{0}})^2 \\
&x^2 + \frac{2}{3}x + \underline{\phantom{0}} = (x\underline{\phantom{0}})^2
\end{align*}
\]

\(^1\)Answers are at the end of the notes.
The last two problems are meant to ensure the reader understands. The key to finding the \( t^2 \) value is to take one-half of the \( b \) value, then square the result.

**Practice Problems 1.** Find the necessary values to create perfect square trinomials.

1. \( x^2 + 4x + \_ = (x \_)^2 \)
2. \( x^2 - 8x + \_ = (x \_)^2 \)
3. \( x^2 + 3x + \_ = (x \_)^2 \)
4. \( x^2 - 5x + \_ = (x \_)^2 \)
5. \( x^2 + 24x + \_ = (x \_)^2 \)
6. \( x^2 + \frac{3}{2}x + \_ = (x \_)^2 \)
7. \( x^2 + 11x + \_ = (x \_)^2 \)
8. \( x^2 - x + \_ = (x \_)^2 \)
9. \( x^2 + 1x + \_ = (x \_)^2 \)
10. \( x^2 - \frac{1}{2}x + \_ = (x \_)^2 \)

1.1.2 Part II

**Example.** Solve \( x^2 + 3x + 1 = 0 \).

This trinomial cannot be factored, as the reader should verify. A prime polynomial indicated no solution earlier; however, consider writing the problem as

\[
x^2 + 3x = -1
\]

Examining only the left hand side (LHS) of the equation, this is the same as the problem in **Practice Problems 1**, # 3. This is the first step in completing the square: moving all constant terms to one side of the equation and any terms with variables to the other side of the equation. The problem begins,

**Solution.**

\[
x^2 + 3x + 1 = 0
\]

\[
x^2 + 3x + \_ = -1
\]

Use \( b = 3 \) to complete the square. Find \( \frac{1}{2}b \) and \( (\frac{1}{2}b)^2 \). This gives \( \frac{3}{2} \) and \( \frac{9}{4} \), respectively. First use \( (\frac{1}{2}b)^2 = \frac{9}{4} \). Add this to both sides of the equation and note that the LHS is now a perfect square and can be factored, which gives

\[
x^2 + 3x + \frac{9}{4} = -1 + \frac{9}{4};
\]

however, it is not necessary to go through the factoring process because this quadratic has been formed by design. At this point \( \frac{1}{2}b = \frac{3}{2} \) is the needed

\^[2]Answers are at the end of the notes
number to write the perfect square of \( x^2 + 3x + \frac{9}{4} = (x + \frac{3}{2})^2 \). This gives,

\[
(x + \frac{3}{2})^2 = -\frac{4}{4} + \frac{9}{4} = \frac{5}{4}
\]

\[
\sqrt{(x + \frac{3}{2})^2} = \sqrt{\frac{5}{4}}
\]

\[
x + \frac{3}{2} = \pm \frac{\sqrt{5}}{2}
\]

\[
x = \frac{\pm \sqrt{5} - 3}{2}
\]

or,

\[
x = \frac{\pm \sqrt{5} - 3}{2}
\]

or,

\[
x = \frac{-3 \pm \sqrt{5}}{2}
\]

**Example.** Solve \( 2x^2 - 5x + 4 = 0 \).

This problem is different than all the preceding problems because the leading coefficient is not 1. *The square cannot be completed until the leading coefficient is 1.* Take time to make sure each step is understood when reading.

**Solution.**

\[
2x^2 - 5x + 4 = 0
\]

\[
2x^2 - 5x = -4
\]

\[
\frac{1}{2}(2x^2 - 5x) = (-4)\frac{1}{2}
\]

\[
x^2 - \frac{5}{2}x = -2
\]

Complete the square of the LHS. Since \( b = -\frac{5}{2} \), then \(-\frac{5}{2} \cdot \frac{1}{2} = -\frac{5}{4}\) and \((-\frac{5}{4})^2 = \)
Then,

\[
x^2 - \frac{5}{2}x + \frac{25}{16} = -2 + \frac{25}{16}
\]

\[
(x - \frac{5}{4})^2 = \frac{-32}{16} + \frac{25}{16}
\]

\[
\sqrt{(x - \frac{5}{4})^2} = \sqrt{\frac{-7}{16}}
\]

\[
x - \frac{5}{4} = \pm \frac{i\sqrt{7}}{4}
\]

\[
x = \frac{\pm i\sqrt{7} + 5}{4}
\]

or,

\[
x = \frac{\pm i\sqrt{7} + 5}{4}
\]

or,

\[
x = \frac{5 \pm i\sqrt{7}}{4}
\]

Homework Problems.

1. \(x^2 + 6x = -5\)  
2. \(x^2 - 4x = 11\)
3. \(x^2 - 10x = -15\)  
4. \(x^2 - 12x = -10\)
5. \(x^2 - 8x = -6\)  
6. \(x^2 + 4x = 3 = 0\)
7. \(x^2 + 10x - 1 = 0\)  
8. \(x^2 + 2x - 1 = 0\)
9. \(x^2 - 2x - 5 = 0\)  
10. \(x^2 - 6x + 6 = 0\)
11. \(x^2 + 12x + 25 = 0\)  
12. \(x^2 - 4x + 2 = 0\)
13. \(2x^2 + 6x - 1 = 0\)  
14. \(2x^2 - 5x + 6 = 0\)
15. \(2x^2 - 2x + 3 = 0\)  
16. \(4x^2 + 2x + 3 = 0\)
Solutions.

Workout. Complete the following perfect squares.

\[ x^2 + 16x + 64 = (x + 8)^2 \]
\[ x^2 - 14x + 49 = (x - 7)^2 \]
\[ x^2 - 10x + 25 = (x - 5)^2 \]
\[ x^2 + 18x + 81 = (x + 9)^2 \]
\[ x^2 - 7x + \frac{49}{4} = (x - \frac{7}{2})^2 \]
\[ x^2 + \frac{2}{3}x + \frac{1}{9} = (x + \frac{1}{3})^2 \]

Practice Problems 1. Find the necessary values to create perfect square trinomials.

1. \( x^2 + 4x + 4 = (x + 2)^2 \)
2. \( x^2 - 8x + 16 = (x - 4)^2 \)
3. \( x^2 + 3x + \frac{9}{4} = (x + \frac{3}{2})^2 \)
4. \( x^2 - 5x + \frac{25}{4} = (x - \frac{5}{2})^2 \)
5. \( x^2 + 24x + 144 = (x + 12)^2 \)
6. \( x^2 + \frac{3}{2}x + \frac{9}{16} = (x + \frac{3}{4})^2 \)
7. \( x^2 + 11x + \frac{121}{4} = (x + \frac{11}{2})^2 \)
8. \( x^2 - x + \frac{1}{4} = (x - \frac{1}{2})^2 \)
9. \( x^2 + 1x + \frac{1}{4} = (x + \frac{1}{2})^2 \)
10. \( x^2 - \frac{1}{2}x + \frac{1}{16} = (x - \frac{1}{4})^2 \)

Homework Problems, Q–2: 1–16

1. \( x = -1, -5 \)
2. \( x = 2 \pm \sqrt{15} \)
3. \( x = 5 \pm \sqrt{10} \)
4. \( x = 6 \pm \sqrt{26} \)
5. \( x = 4 \pm \sqrt{10} \)
6. \( x = -2 \pm \sqrt{7} \)
7. \( x = \pm \sqrt{26} - 5 \)
8. \( x = -1 \pm \sqrt{2} \)
9. \( x = 1 \pm \sqrt{6} \)
10. \( x = 3 \pm \sqrt{3} \)
11. \( x = \pm \sqrt{11} - 6 \)
12. \( x = 2 \pm \sqrt{2} \)
13. \( x = -\frac{3}{2} \pm \frac{\sqrt{11}}{2} \)
14. \( x = \frac{5 \pm i\sqrt{23}}{4} \)
15. \( x = \frac{1 \pm i\sqrt{5}}{2} \)
16. \( x = \frac{-1 \pm i\sqrt{11}}{4} \)